Aside: Second Fundamental Theorem of Calculus: (2FTC)

If $F(x)$ is continuous and a is a constant, then if we let $G(x) = \int_a^x F(t)\,dt$, then

$$G'(x) = F(x).$$

Upshot: This gives a (strange) recipe for constructing an antiderivative of *any* function, even if we can't find a formula for it.

By approximating the integral, we can find a numerical antiderivative of the function.

Ⓔⓧ ① Let $B(x) = \int_{-\pi/2}^{x} \cos(t)\,dt$.

By 2FTC,

$$B'(x) = \cos(x).$$

Let's check for fun: $B(x) = \sin(t)\Big|_{-\pi/2}^{x} = \sin(x) - \underbrace{\sin\left(-\tfrac{\pi}{2}\right)}_{-1}$

$\Rightarrow B(x) = \sin(x) + 1$

$\Rightarrow B'(x) = \cos(x)$. ✓

② Let $g(y) = \int_0^y e^{-x^2}\,dx$.

By 2FTC, $g'(y) = e^{-y^2}$

This fcn $g(y)$ is important in statistics.
No other way to calculate $g'(y)$.

③ Let $f(x) = \int_{x^2}^{2} \sin(b^2)\,db$

What is $f'(x)$?

$f(x) = -\int_2^{x^2} \sin(b^2)\,db$

Use chain rule

If: $g(u) = -\int_2^{u} \sin(b^2)\,db$

$g'(u) = -\sin(u^2)$

$\therefore f'(x) = -\sin\left((x^2)^2\right)\cdot(x^2)'$

$= \boxed{-2x\sin(x^4)}$.

④ let $F(t) = \int_{t^2}^{t^3} e^{u^2}\,du$. $F'(t) = ?$

Note $F(t) = \int_{t^2}^{0} e^{u^2}\,du + \int_0^{t^3} e^{u^2}\,du$

Property of integrals: $\int_a^b \sim + \int_b^c \sim = \int_a^c \sim$.

$$\Rightarrow F(t) = -\int_0^{t^2} e^{u^2}\,du + \int_0^{t^3} e^{u^2}\,du$$

$$F'(t) = -e^{(t^2)^2}\cdot(2t) + e^{(t^3)^2}\cdot 3t^2$$

$$= -2te^{t^4} + 3t^2e^{t^6}.$$

Time for more integration techniques

....

- Trigonometric integrals

Warm-up:

① $\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + C$

② $\int \cos^2(3x)\,dx = \int\left(\frac{1}{2} + \frac{1}{2}\cos(6x)\right)dx$

$$= \boxed{\frac{1}{2}x + \frac{1}{12}\sin(6x) + C}$$

- $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$
- $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$

③ $\int \cos^3(3x)\,dx = \int \cos^2(3x)\cdot\cos(3x)\,dx$

$$= \int \left(1-\sin^2(3x)\right)\underline{\cos(3x)\,dx}$$

Subs: let $u = \sin(3x)$, $du = 3\cos(3x)\,dx$

$$\underline{\tfrac{1}{3}du} = \underline{\cos(3x)\,dx}$$

$$= \int (1-u^2)\tfrac{1}{3}\,du = \int \left(\tfrac{1}{3} - \tfrac{1}{3}u^2\right) du$$

$$= \frac{u}{3} - \frac{1}{3}\,\frac{1}{3}u^3 + C$$

$$= \boxed{\frac{1}{3}\sin(3x) - \frac{1}{9}\sin^3(3x) + C}$$

---

④ $\int \cos^4(3x)\,dx = ?$

$$= \int \underbrace{\cos^2(3x)}_{\frac{1}{2}(1+\cos(6x))}\underbrace{\cos^2(3x)}_{\frac{1}{2}(1+\cos(6x))}\,dx$$

$$= \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(6x)\right)\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(6x)\right)dx$$

$$= \frac{1}{4}\int (1+\cos(6x))(1+\cos(6x))\,dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos(6x) + \cos^2(6x)\right)dx$$

$$= \underline{\tfrac{1}{4}}\int \left(\underline{1} + 2\cos(6x) + \underline{\tfrac{1}{2}} + \underline{\tfrac{1}{2}}\cos(12x)\right)dx$$

$$= \int \left[\frac{3}{8} + \frac{1}{2}\cos(6x) + \frac{1}{8}\cos(12x)\right] dx$$

$$= \boxed{\frac{3}{8}x + \frac{1}{12}\sin(6x) + \frac{1}{96}\sin(12x) + C}$$ ✓

⑤ $\int \cos^5(3x)\,dx$ ← how would we calculate this?

$$= \int \underbrace{\cos^2(3x)}_{1-\sin^2} \underbrace{\cos^2(3x)}_{1-\sin} \underline{\cos(3x)}\,dx$$

$$= \int \left(1-\sin^2(3x)\right)^2 \underline{\cos(3x)\,dx}$$

let $u = \sin(3x) \Rightarrow du = 3\cos(3x)dx$

$\frac{1}{3}du = \cos(3x)\,dx$

$$= \int (1-u^2)^2 \frac{1}{3}du = \int \frac{1}{3}\left(1-2u^2+u^4\right) du$$

$$= \int \left(\frac{1}{3} - \frac{2}{3}u^2 + \frac{1}{3}u^4\right) du$$

$$= \frac{u}{3} - \frac{2}{9}u^3 + \frac{1}{15}u^5 + C$$

$$= \boxed{\frac{\sin(3x)}{3} - \frac{2}{9}\sin^3(3x) + \frac{1}{15}\sin^5(3x) + C}$$

⑥ $\int \sec^4(\theta)\, d\theta = ?$

$= \int \frac{1}{\cos^4(\theta)}\, d\theta$

$\sin^2\theta + \cos^2\theta = 1$

div by $\cos^2(\theta)$

$\tan^2(\theta) + 1 = \sec^2\theta$

$= \int \sec^2(\theta)\sec^2(\theta)\, d\theta$

$= \int (\tan^2(\theta) + 1)\sec^2(\theta) d\theta$

let $u = \tan(\theta)$, $du = \sec^2(\theta) d\theta$

$= \int (u^2+1)\, du = \frac{u^3}{3} + u + C$

$= \boxed{\frac{\tan^3(\theta)}{3} + \tan\theta + C}$

⑦ $\int \tan^2(\theta) d\theta = \int (\sec^2\theta - 1)\, d\theta$

$= \boxed{\tan\theta - \theta + C}$

⑧ $\int_0^{\pi/4} \tan(\theta) d\theta = \int_0^{\pi/4} \frac{\sin(\theta)}{\cos(\theta)} d\theta$

let $u = \cos(\theta) \Rightarrow du = -\sin(\theta) d\theta$

$-du = \sin(\theta) d\theta$

$= \int_{u=\cos(0)=1}^{1/\sqrt{2}} \frac{-du}{u}$

$$= -\int_1^{1/\sqrt{2}} \frac{1}{u}\,du = -\ln|u|\,\Big|_1^{1/\sqrt{2}}$$

$$= -\ln\left|\frac{1}{\sqrt{2}}\right| + \underset{0}{\ln|1|}$$

$$= -\ln\left(2^{-1/2}\right)$$

$$= -\left(-\frac{1}{2}\right)\ln(2) = \boxed{\frac{1}{2}\ln 2}$$

area.

---

⑨ $$\int \frac{e^{15t}\,dt}{e^{16t} + 4e^{14t}} = ?$$

$$= \int \frac{\cancel{e^{14t}} \cdot e^{t}}{\cancel{e^{14t}}\left(e^{2t}+4\right)}\,dt$$

let $u = e^{t}$ $du = e^{t}\,dt$

$$= \int \frac{du}{u^2+4}$$

let $u = 2w$ $du = 2\,dw$

$$= \int \frac{2\,dw}{4w^2+4} = \frac{1}{2}\int \frac{1}{w^2+1}\,dw$$

$$= \frac{1}{2}\arctan(w) + C$$

$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$$

$$= \boxed{\frac{1}{2}\arctan\left(\frac{e^t}{2}\right) + C}.$$

---

⑩ $$\int \sec(a)\, da$$

$$\cancel{\frac{1}{\sec(a)}\sec^2(a)\, da}$$

$$\frac{\sin^2 + \cos^2 = 1}{\cos^2}$$

$$\tan^2 + 1 = \sec^2$$

$$\sqrt{\tan^2+1} = \text{se-}$$

$$\int \sec(a) \cdot \frac{\sec(a) + \tan(a)}{\sec(a) + \tan(a)}\, da$$

$$= \int \frac{\sec^2(a) + \sec(a)\tan(a)}{\sec(a) + \tan(a)}\, da$$

let $u = \sec(a) + \tan(a)$

$du = (\sec(a)\tan(a) + \sec^2(a))\, da$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \boxed{\ln|\sec(a) + \tan(a)| + C}$$

Another way to do the same integral:

$$\int \sec(a)\,da = \int \frac{1}{\cos(a)}\,da = \int \frac{\cos(a)\,da}{\underset{1-\sin^2}{\cos^2(a)}}$$

$$= \int \frac{\cos(a)\,da}{1-\sin^2(a)}$$

let $u = \sin(a) \Rightarrow du = \cos(a)\,da$

$$= \int \frac{du}{1-u^2} = \int \frac{1}{(1-u)(1+u)}\,du$$

$$= \int \left(\frac{\frac{1}{2}}{1-u} + \frac{\frac{1}{2}}{1+u} \right) du$$

check: $\frac{\frac{1}{2}(1+u)}{(1-u)(1+u)} + \frac{\frac{1}{2}(1-u)}{(1-u)(1+u)} = \frac{\frac{1}{2}+\frac{1}{2}u+\frac{1}{2}-\frac{1}{2}u}{(1-u)(1+u)}$

$= \frac{1}{(1-u)(1+u)}$

$$= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C$$

$$= \boxed{-\frac{1}{2}\ln|1-\sin(a)| + \frac{1}{2}\ln|1+\sin(a)| + C}$$

Extra Credit ↗ why are these two answers the same?